Effect of Hydrodynamic Instabilities on the Migration of a Solid Particle

Blaise NSOM, Noureddine LATRACHE, Ahmad AHMAD & Clément BERNARD

Université Européenne de Bretagne – Université de Brest.
LBMS-EA 4325. LIME/IUT de Brest. BP 93169. Rue de Kergoat. 29231 BREST Cedex 3 (France)

Corresponding author: Blaise.nsom@univ-brest.fr

Abstract
The present paper aims at modeling the turbulent migration of spherical rigid particles in a Newtonian shear flow. The turbulent regime being itself still an open field of Fluid Dynamics research, we assumed the transient regime to represent the main characteristics of the physics of the particles migration in shear flow. First of all, expressions of the velocity field of the perturbation were obtained analytically. Then, the equations of motion of the solid particle were written and integrated. The difference between the trajectories in the disturbed flow and in the basic flow represents the turbulent dispersion of the particle. The Couette configuration was selected as its transient regime, called Taylor vortex flow is a classical problem of Fluid Dynamics..

Keywords: Instability, Integral method, Particle migration, Spherical harmonics, Spherical particle, Streamlines, Taylor-Couette flow, Trajectory, Transient regime, Turbulence.

1- Background

The knowledge of the distribution of the solid particles in a turbulent suspension flow is a key issue of Mechanical Engineering and Environmental Engineering, as well. This secular problem has been tackled by many authors, both experimentally and numerically. The numerical studies use Lagrangian or Eulerian. The Lagrangian approach simulate a number of individual particle trajectories in a given turbulence field and derive the particle behaviour from the mean values obtained from statistical computations. While the Eulerian approach assume that two continuous fields: the particle field and the suspending liquid field coexist and transport equations are solved in both phases. The experimental studies generally consider the time variation of the spreading distance. Unfortunately, an unconciliable discrepancy exists between the reported scaling laws. The method generally consists in tracking the particle using an appropriate technique (image analysis…), [1].

When the basic flow is known, analytical expressions of the velocity field around the particle can be calculated. For example, the velocity field around a spherical particle with radius b embedded in an inviscid uniform flow U_∞ is given in a spherical $(r,\theta\varphi)$ system of coordinates, by

$$\vec{V}=U_\infty\left(1-\frac{b^3}{r^3}\right)\cos(\theta)\vec{e}_r-U_\infty\left(1+\frac{b^3}{r^3}\right)\sin(\theta)\vec{e}_\theta \qquad (1)$$

and the streamlines can easily be drawn as shown in figure 1, and this is the method used here for the Couette flow both in the laminar regime and in the Taylor vortex flow, itself representing the turbulent regime.

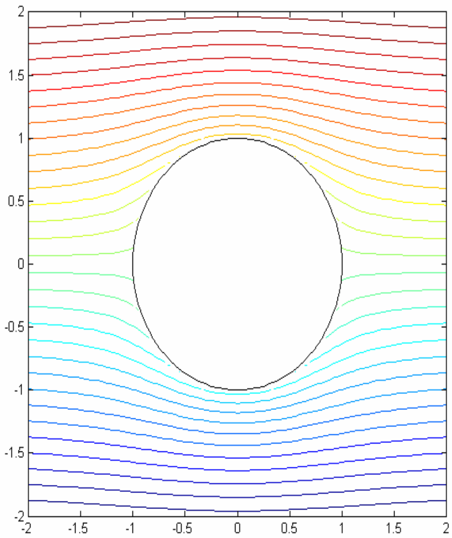

Figure 1: Streamlines around a solid sphere in inviscid fluid with uniform velocity at infinity

When the fluid is viscous, the velocity flied takes the form

$$V = U_\infty \cos(\theta)\left[1 - \frac{3b}{2r} + \frac{b^3}{2r^3}\right]\vec{e}_r - U_\infty \sin(\theta)\left[1 - \frac{3b}{4r} - \frac{b^3}{4r^3}\right] \qquad (2)$$

The streamlines are shown in figure 2 and they agree with Schlichting [2].

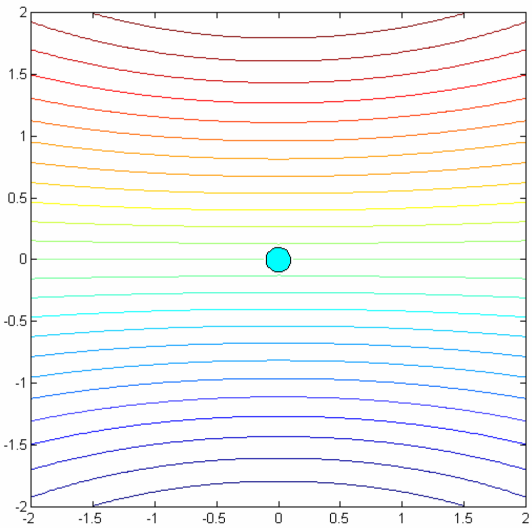

Figure 1: Streamlines around a solid sphere in viscous fluid with uniform velocity at infinity

This is the method which we used in the present work for the Couette flow, which is generated in a viscous fluid contained between two concentric cylinders, with the inner cylinder rotating and the outer cylinder being kept at rest (fig. 4)

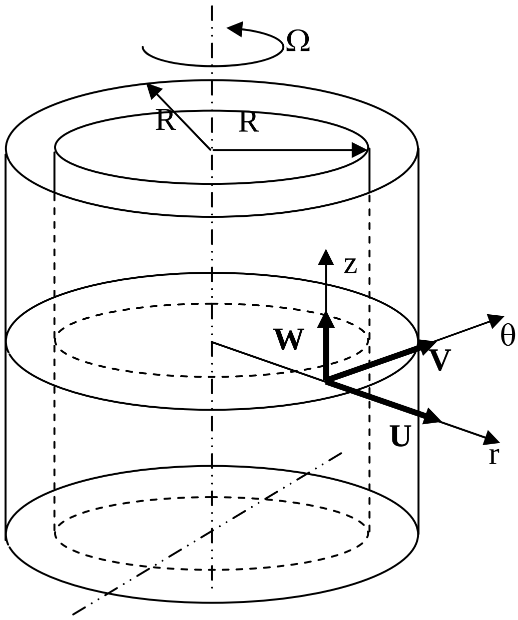

Fig. 3: The Couette-Taylor system

At low Reynolds number, a circular Couette flow exists. If the angular speed is increased and exceeds a critical given value, the flow develops a Taylor vortex flow made of toroidal contrarotative cells [3, 4]. This secondary flow which is periodic and periodic along the cylinders axis and is due to the competition between the viscous and the centrifugal forces. It is the first step of the flow transient regime and it can describe a first insight of the phenomena developping in the turbulent regime [5]. The cylinders radii are much larger than the gap in the limit where a plane configuration can be used. In this operation, the (fixed) cylindrical system of coordinates (r,θ,z) becomes cartesian (x',y',z') and the velocity field of the resulting plane Couette flow is written (fig. 4)

$$\vec{V}=Ax'\vec{e}_{y'} \tag{4}$$

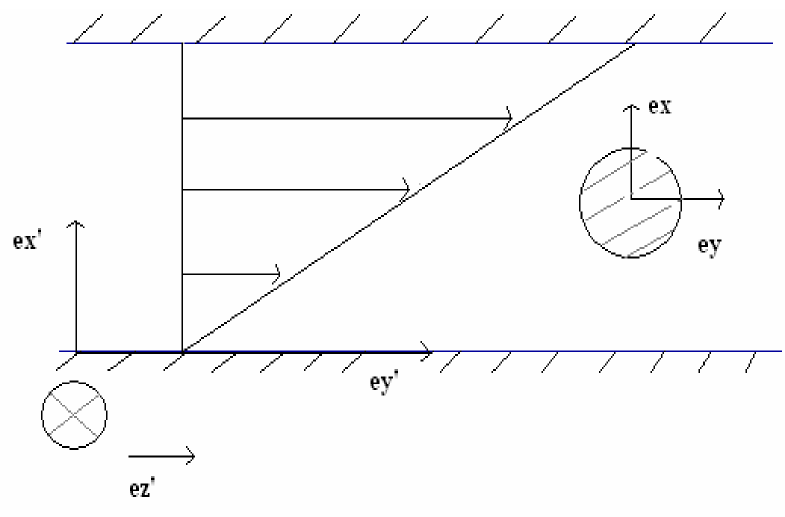

Figure 4: The plane Couette flow

Moreover, the particle which has the same density as the fluid, is placed at the mid-gap has a radius much smaller than the device gap so that its interactions with the solid walls can be neglected.

2- Velocity flield around a particle in basic Couette flow

To calculate the perturbation to the latter basic Couette flow, the velocity field described by eq. (4) is broken down in two components, say a symmetric elementary velocity field $\vec{V}_1$ and an anti-symmetric one $\vec{V}_2$ respectively given by:

$$\vec{V}_1=\frac{A}{2}y'\vec{e}_x+\frac{A}{2}x'\vec{e}_y \quad ; \quad V_2=-\frac{A}{2}y'\vec{e}_x+\frac{A}{2}x'\vec{e}_y \qquad (5)$$

The perturbation to $\vec{V}_1$ is given by Landau and Lifchitz [6], and the perturbation to $\vec{V}_2$ is given by Fortier [7]. With respect to a cartesian system of coordinates (x,y,z) attached to the particle, the superposition of the two previous perturbations leads to the following solution:

$$\vec{V}'_x=A\left[\frac{5}{2}x^2y\left(\frac{b^5}{r^7}-\frac{b^3}{r^5}\right)-\frac{y}{2}\left(\frac{b^5}{r^5}-\frac{b^3}{r^3}\right)\right]\vec{e}_x+A\left[\frac{5}{2}xy^2\left(\frac{b^5}{r^7}-\frac{b^3}{r^5}\right)-\frac{x}{2}\left(\frac{b^5}{r^5}-\frac{b^3}{r^3}\right)\right]\vec{e}_y+\frac{5}{2}Axyz\left(\frac{b^5}{r^7}-\frac{b^3}{r^5}\right)\vec{e}_z \qquad (6)$$

where $r=\sqrt{(x^2+y^2+z^2)}$, while the pressure field is given by the equation of Stokes.

Note that, apart the translation described by eq. (6), the particle undergoes a rotation around $\vec{e}_z$ with angular speed $\Omega=\frac{A}{2}$, so that the resulting velocity field of the perturbation is given by

$$\vec{V}'=A\left[\frac{5}{2}\left(\frac{b^5}{r^7}-\frac{b^3}{r^5}\right)x^2y-\frac{1}{2}\frac{b^5}{r^5}y\right]\vec{e}_x+A\left[x+\frac{5}{2}\left(\frac{b^5}{r^7}-\frac{b^3}{r^5}\right)y^2x-\frac{1}{2}\frac{b^5}{r^5}x\right]\vec{e}_y+\frac{5}{2}A\left(\frac{b^5}{r^7}-\frac{b^3}{r^5}\right)xyz\vec{e}_z \qquad (7)$$

The corresponding streamlines show that the flow is radial far from the particle and that the streamtubes are circular cones. Furthermore, for a particle placed at a position $M(x_M,y_M,z_M)$ different from the cell axis but far from the solid walls, the previous expressions remain valid provided that x,y,z be replaced by $(x-x_M)$, $(y-y_M)$ and $(z-z_M)$ respectively.

3- Velocity field of Taylor-vortex flow in the absence of the particle

To compute the perturbation caused by the presence of the particle in the transient regime, we need the exact solution of the velocity field of the Taylor-Couette flow.

a- Equations

The Couette-Taylor flow remains a classical teaching tool of the Fluid Dynamics Stability topic and it has been the subject of numerous research papers in different configurations (e.g.: [3]). The velocity field of the circular Couette flow has the following expression:

$$V=Ar+\frac{B}{r} \quad ; \quad A=-\Omega\frac{\eta^2}{1-\eta^2} \quad ; \quad B=\Omega R_1^2\frac{1}{1-\eta^2} \quad ; \quad U=0 \quad ; \quad W=0 \qquad (8)$$

The Taylor vortex flow is described by the superposition of a small perturbation of the form:

$$\underline{v_r}(r,z)=\bar{u}(r,z)=u'(r)\cdot\cos(kz) \quad ; \quad \underline{v_\theta}(r,z)=\bar{v}(r,z)=v'(r)\cdot\cos(kz) \qquad (9)$$

$$\underline{v_z}(r,z)=\bar{w}(r,z)=w'(r)\cdot\sin(kz) \quad ; \quad \bar{p}(r,z)=p'(r)\cdot\cos(kz) \qquad (10)$$

on circular Couette flow. Applying the equation of continuity to the resultant motion leads to

$$w'(r)=-\frac{1}{k}\cdot\frac{du'}{dr} \qquad (11)$$

Define the non-dimensional velocity field by

$$u(r,z) = R_1\omega \cdot u' \cdot \cos\left(\frac{kz}{R_1}\right) \; ; \; v(r,z) = R_1\omega \cdot v' \cdot \cos\left(\frac{kz}{R_1}\right) \; ; \; w(r,z) = R_1\omega \cdot w' \cdot \sin\left(\frac{kz}{R_1}\right) \quad (12)$$

Let

$$\delta = \frac{R_2}{R_1} - 1 \quad (13)$$

The flow dynamics can be described by the Taylor number defined as

$$T = \frac{2(Re)^2}{\delta} \quad (14)$$

where Re denotes the Reynolds number, itself defined as

$$Re = \frac{\rho \cdot \delta^2 \cdot R_1^2 . \omega}{\mu} \quad (15)$$

Linearizing the Navier Sokes equations for the superposition of the basic Couette flow with the previous perturbation, leads to the following equation in the non dimensional form:
equation for v :

$$\left[D^6 - A_4 D^4 + A_2 D^2 - A_0 - B \cdot x\right] v(x) = 0 \quad (16)$$

and which is completed by the no-slip condition

$$u(x) = 0 \quad ; \quad v(x) = 0 \quad ; \quad \frac{du(x)}{dx} = 0 \text{ at } x = 0 \text{ and } 1 \quad (17)$$

where $D^i v(x)$ denotes the i^{th} derivative of $v(x)$ and the coefficients A_i's and B are defined by

$$A_4 = 3a^2 \qquad A_2 = 3a^4 \qquad B = Ta^2 \qquad A_0 = a^6 - B \quad (18)$$

b- Solution

A solution of (14) can be found in integral form, i.e. such that

$$v(x) = \int_r^{\varepsilon} e^{xt} v(t) \cdot dt \quad (19)$$

So, equ. (14) becomes

$$\int_r^{\varepsilon} \left(t^6 - A_4 t^4 + A_2 t^2 - A_0\right) e^{xt} v(t) \cdot dt - B \int_r^{\varepsilon} x \cdot e^{xt} v(t) \cdot dt = 0 \quad (20)$$

which can be written in the form

$$\int_\gamma^{\varepsilon} \left[\left(t^6 - A_4 t^4 + A_2 t^2 - A_0\right) e^{xt} v(t) + B e^{xt} \frac{dv(t)}{dt}\right] dt - C = 0 \qquad \text{where} \quad C = B \int_\gamma^{\varepsilon} \frac{\partial\left[e^{xt} v(t)\right]}{\partial t} dt \quad (21)$$

Let us now search v(t) such that

$$\left(t^6 - A_4 t^4 + A_2 t^2 - A_0\right) v(t) + B \cdot \frac{dv(t)}{dt} = 0 \quad (22)$$

This is a first order differential equation for *v(t)* with solution

$$v(t) = C \cdot \exp\left[-\frac{t^7}{7B} + \frac{A_4 t^5}{5B} - \frac{A_2 t^3}{3B} + \frac{A_0 t}{B}\right] \quad (23)$$

Then, the solution of the equation of motion is given by

$$v(x) = \int_r^{\varepsilon} e^{xt} \exp\left[-\frac{t^7}{7B} + \frac{A_4 t^5}{5B} - \frac{A_2 t^3}{3B} + \frac{A_0 t}{B}\right] dt \quad (24)$$

provided that

$$B\int_{r}^{\varepsilon}\left[\frac{\partial\left(e^{xt}v(t)\right)}{\partial t}\right]\cdot dt=0 \qquad (25)$$

The integral boundaries are chosen such that

$$e^{xt}\cdot\exp\left[-\frac{t^7}{7B}+\frac{A_4t^5}{5B}-\frac{A_2t^3}{3B}+\frac{A_0t}{B}\right]=0 \qquad (26)$$

In the complex plane of variable t, any contour such that, at its ends,

$$Real\left\{-\frac{t^7}{7}+\frac{A_4t^5}{5}-\frac{A_2t^3}{3}+A_0t\right\}=-\infty \qquad (27)$$

is satisfactory. Since the above polynomial behaves at infinity as $\left(-\frac{t^7}{7}\right)$, the three contours drawn in figure 5 are suitable.

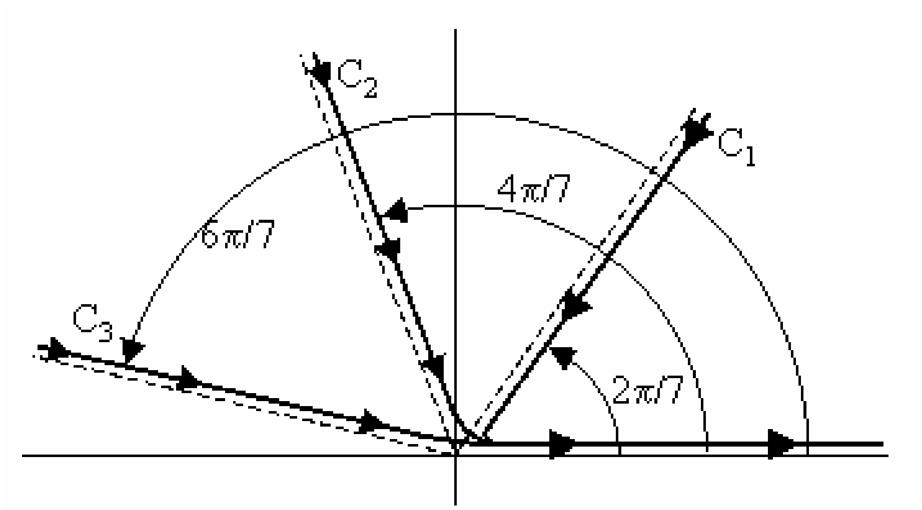

Figure 5: The integration contours in the complex plane of variable t.

The solution of the equation of motion (22) is a linear combination of the previous six particular solutions. The infinite integrals appearing in this solution are computed numerically and their convergence is rapid. The marginal stability curve obtained is presented on fig.6 and the following critical parameters are derived: critical Taylor number: $T_c=3389$ and critical wave number: $a_c=3.12$. These results are in an excellent agreement with the classical ones obtained experimentally or theoretically [4].

4- Perturbation due to the particle on the Taylor vortex flow

Returning to the approximation of the plane configuration and assuming a creeping flow with no wake behind the particle, the hydrodynamic field $(\vec{v}'_1, p'_1)$ can be described by the Stokes equation together with the continuity equation:

$$\vec{\nabla}p'_1=\mu\Delta\vec{v}'_1 \quad ; \quad \nabla\vec{v}'_1=0 \qquad (28)$$

and the non slip condition

$$\vec{v}+\vec{v}'_1=\vec{0} \quad \text{on the solid walls} \qquad (29)$$

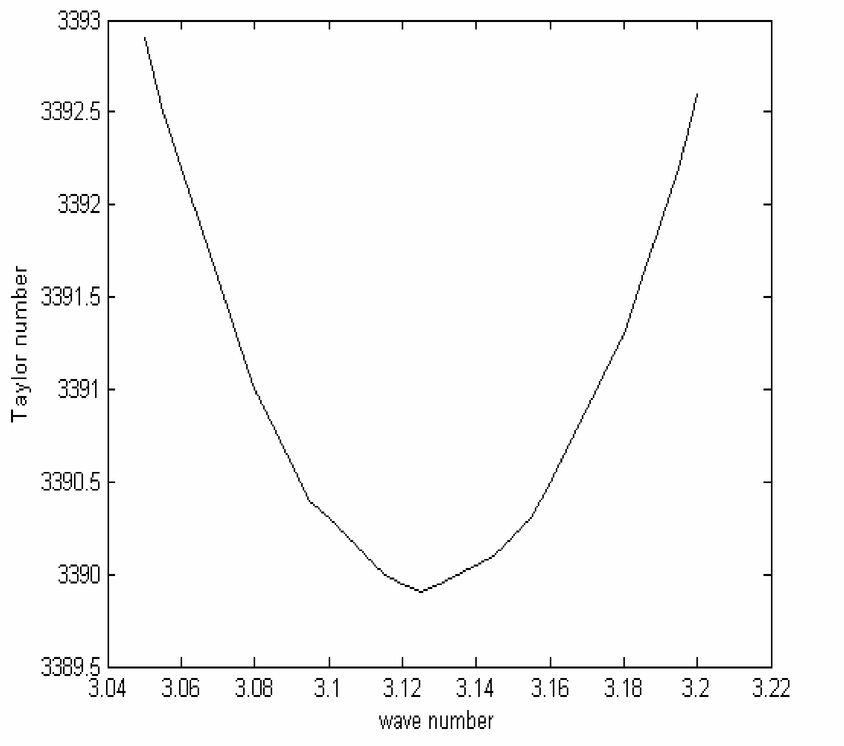

Fig.6: The marginal stability curve of a Couette-Taylor flow of a homogeneous fluid with the narrow gap approximation.

Remark that the Stokes equations are the only ones which can provide a translation/rotation creeping motion on the sphere while the solution of the Navier-Stokes equations lead to secondary motions which are centrifuge in the vicinity of the equator and centricentripetal at the poles.

This problem admits an exact solution an exact solution in the integral form with the mean of the Green tensor [8, 9]. The solution can also be expanded using spherical harmonics [10, 11], as follows:

a- On the particle, the velocity field of the secondary motion is expanded in spherical harmonics (in the usual spherical system of coordinates) as:

$$\frac{\vec{r}}{r}\cdot\vec{v}=\sum_{n=1}^{\infty}X_n \quad ; \quad -r\vec{\nabla}\cdot\vec{v}=\sum_{n=1}^{\infty}Y_n \quad ; \quad r\nabla\vec{v}=\sum_{n=1}^{\infty}Z_n \qquad (30)$$

b- The spherical harmonics p_n,Φ_n,Π_n are then formed using the following relations:

$$p_{-(n+1)}=\frac{\mu(2n-1)}{(n+1)}\cdot\frac{1}{b}\left(\frac{b}{r}\right)^{n+1}\left[(n+2)X_n+Y_n\right] \quad ; \quad \Phi_{-(n+1)}=\frac{1}{2(n+1)}b\left(\frac{b}{r}\right)^{n+1}\left[nX_n+Y_n\right] \qquad (31)$$

$$\Pi_{-(n+1)}=-\frac{1}{n(n+1)}\left(\frac{b}{r}\right)^{n+1}Z_n \quad \text{where} \quad \mu=\rho\nu \qquad (32)$$

A detailed presentation on the theory of spherical harmonics is provided in [12, 13].

c- The perturbation sought is then given by

$$\vec{v}'=\sum_{n=1}^{\infty}\left[\vec{\nabla}\wedge\left(\vec{r}\Pi_{-(n+1)}\right)+\vec{\nabla}\Phi_{-(n+1)}-\frac{(n-2)}{\mu\cdot 2n(2n-1)}r^2\vec{\nabla}p_{-(n+1)}+\frac{(n+1)}{\mu\cdot n(2n-1)}\vec{r}p_{-(n+1)}\right] \quad (33)$$

After algebraic a careful handling, the terms of the 1st order of the expansion in spherical harmonics are obtained with the following expressions

$$X_1=\frac{3u\cos\theta}{2}\int_0^{\pi}\sin^2\theta'\cdot\cos\theta'\cdot d\theta=0 \quad (34)$$

$$Y_1=-2X_1=0 \quad (35)$$

$$Z_1=\frac{3v\cos\theta}{2}\int_0^{\pi}\sin^2\theta'\cdot d\theta'=\frac{3\pi v\cdot\cos\theta}{4} \quad (36)$$

After calculating X_2, Y_2, Z_2 the expressions for higher orders were vanishing and we finally obtained the following expression for the velocity field induced by the presence of the particle on the Taylor vortex flow:

$$v'=\frac{5\pi u}{128}\left[\left(\frac{b}{r}\right)^4\left(3\sin^2(\theta)+44\right)+\left(\frac{b}{r}\right)^2\left(3\sin^2(\theta)-64\right)\right]\vec{e}_r-\frac{5\pi u}{64}\left(\frac{b}{r}\right)^4\sin(\theta)\cos(\theta)\vec{e}_\theta+\frac{3\pi v}{8}\left(\frac{b}{r}\right)^2\cos(\theta)\vec{e}_z \quad (37)$$

This solution verifies the non slip condition on the cylinders walls, as its different terms are proportional to the velocity field of the secondary motion in the absence of the sphere (Taylor vortex flow). Moreover, the solution vanishes at the infinity. Therefore, it can represent the perturbation sought. With the assumptions of a creeping stationary flow around the particle, the sphere trajectory can be now easily computed in the cell and its migration characterized. Finally, for an assembly of particles, the induced velocity field can be obtained by applying the mathod of images on the previous solution.

References

[1] Latrache, N., Ahmad, A. & Nsom, B.: The propagation of pollutant with fresh water and a polyethyleneoxide polymer solution in salt water, Proceedings of the “Safer Seas” International Conference, Brest (France), October 2007 – 5 pages on CD.
[2] Schlichting, H.: “Boundary-Layer Theory”, Mc Graw-Hill, 6th eds., New-York, 1968
[3] Jenny, M. & Nsom, B. : Primary instability of a Taylor-Couette flow with a radial stratification and radial buoyancy, Phys. Fluids, 19, 108104, 2007
[4] Chandrasekhar, S.: Hydrodynamic and hydromagnetic stability. Dover Publication, Inc. New York (1961).
[5] Nsom, B.: “Computation of drag reduction in fiber suspensions”, Fluid Dyn. Res., 14, 275-288, 1994
[6] Landau, L. & Lifchitz, .: “Mécanique des fluids”, eds. MIR (translated from Russian in French), 1971
[7] Fortier, A. “Mécanique des suspensions”, Masson, Paris (1967)
[8] Flugge, S.& Truesdell, C.: „Fluid Dynamics II“, Springer, Berlin, 1963
[9] Kanwall, R.P.: „Linear integral equations. Theory and technique”, Academic Press, New-York, 1971
[10] Happel, J. & Brenner, H.: “Low Reynolds number hydrodynamics”, PrenticeHall Inc., 1965
[11] Lamb, H.: “Hydrodynamics”, Dover – 6th eds., New-York, 1932
[12] Hosbon, E.W.: “The theory of spherical and ellipsoidal harmonics”, Cambridge University Press, 1931
[13] Robin, L.: “Fonctions sphériques de Legendre et functions sphéroïdales », Gauthier-Villars, Paris, 1957 (in French)