A coupled-mode approach to nonlinear waves in finite depth. Viscous bottom boundary-layer flow.

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ABSTRACT

A weakly dissipative free-surface flow model is presented, based on the potential flow approach previously developed by the authors (Athanassoulis & Belibassakis 2002, 2006). The potential flow model is derived with the aid of Luke's (1967) variational principle, in conjunction with a complete vertical expansion, leading to a non-linear coupled-mode system of horizontal equations. The latter coupled-mode system models the evolution of nonlinear water waves over a general bathymetry in intermediate and shallow water depth conditions. The consistent coupled-mode system has been applied to numerical investigation of families of steady travelling wave solutions in constant depth (Athanassoulis & Belibassakis 2007) showing good agreement with known solutions both in the Stokes and the cnoidal wave regimes. In the present work, the above coupled-mode model is linked with laminar bottom boundary layer equations, permitting the investigation of viscous effects on wave propagation up to leading-order.

1 INTRODUCTION

We consider the problem of non-linear gravity waves propagating over a general bathymetry (see Fig.1). An important feature of this problem is that propagation phenomena take place in horizontal directions, and non-local couplings (wave-wave and wave-seabed) exist through the vertical structure of the flow field. Extra difficulties are introduced by the fact that no asymptotic assumptions concerning the free-surface and bottom slope are made. Ignoring the effects of viscosity at a first stage, and using Luke’s (1967) variational principle, in conjunction with an enhanced local-mode series expansion of the wave potential developed by the authors (Athanassoulis & Belibassakis 2002, Belibassakis & Athanassoulis 2006), a non-linear coupled-mode system of equations on the horizontal plane has been derived, modelling the evolution of nonlinear water waves in intermediate depth and over a general bathymetry. The vertical structure of the wave field is exactly represented by means of a local-mode series expansion of the wave potential. This series contains the usual propagating and evanescent modes, plus two additional terms, the free-surface mode and the sloping-bottom mode, enabling the consistent treatment of the non-vertical end-conditions at the free-surface and the bottom boundaries.

One of the characteristic effects of viscosity on free oscillatory flow in intermediate and shallow water-depth is the generation of bottom boundary layer and the damping of water waves, that could become important over long time compared to the typical period and/or long distance compared to the typical wavelength, Mei (1983). A simplified approach to account for dissipation effects due to viscosity in ideal-flow models is through a damping term in the propagation equations or the depth-integrated momentum equations, based on a frictional empirical coefficient; see, e.g., Dingemans (1997). This term is taken to be proportional to the irrotational horizontal bottom velocity, leading to the incorrect result that the shear stress is in phase with the irrotational velocity near the bottom and the model has no memory associated with the history of the oscillatory flow. In enhanced models able to predict water wave transformation from deep water to the nearshore/coastal area and over a general bathymetry (characterized by possibly steep bottom slope), it is essential to have an accurate estimation of the bottom shear stress. This will enhance the predictive capability of the above wave models as concerns viscous damping and, especially, concerning the detailed structure of the flow in the oscillating boundary layer. The latter is quite important for estimating mass transport near the sea bottom, finding thus, useful applications to sediment transport studies.

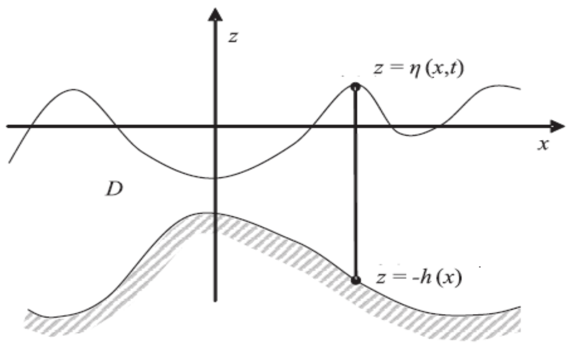

Fig 1. Water waves propagating over a generally-shaped (nonuniform) strip

In this direction, Liu & Orfila (2004), using a perturbation approach and the Boussinesq approximation, derived sets of depth-integrated continuity and momentum equations for transient long-wave propagation including viscous effects. The resulting equations are differential–integral equations in terms of the depth-averaged horizontal velocity (or velocity evaluated at certain depth) and the free-surface displacement, in which the viscous terms are represented by convolution integrals. The latter model has been used by Liu et al (2007) to examine the boundary layer flow characteristics under a solitary wave. In the same line of approach, Dutykh & Dias (2007a) derived and studied new models proposed to add dissipative effects in the context of the Boussinesq equations, which include the effects of weak dispersion and nonlinearity in a shallow water framework. Extension of these ideas to 3D weakly dissipative free-surface flows are presented in Dutykh & Dias (2007b) using the classical potential flow approach. In this work, the coupled-mode system developed by the authors is linked with laminar bottom boundary layer equations, permitting the investigation of viscous effects on wave propagation up to leading-order. It is shown that the present model represents well the structure of laminar bottom boundary layer, permitting more accurate estimation of viscous damping of progressive waves in intermediate and shallow water depth conditions.

2 THE NONLINEAR COUPLED-MODE SYSTEM (CMS)

We restrict ourselves to the 2D problem corresponding to normally incident waves propagating in a general (non-uniform) strip D, bounded below by the seabed $z=-h(x)$, and above by the free surface $z=\eta(x,t)$; see Fig.1. Luke's functional, modelling the inviscid, homogeneous, nonlinear water-wave problem, is defined by the integration of pressure in *D*:

$$F[\Phi,\eta]=\int_{t_1}^{t_2}\int_{x_1}^{x_2} dxdt \int_{z=-h(x)}^{z=\eta(x,t)} \left\{ \frac{\partial\Phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right\}+gz\right\}dz \quad , \tag{1}$$

where $\Phi=\Phi(x,z,t)$ is the velocity potential. The nonlinear water-wave problem is equivalently formulated by the variational equation:

$$\delta F[\Phi,\eta]=\delta_\Phi F[\Phi,\eta]+\delta_\eta F[\Phi,\eta]=0 \tag{2}$$

where the first variation of $F[\Phi,\eta]$ is obtained as the sum of its partial variations with respect to the fields $\Phi=\Phi(x,z,t)$ and $\eta=\eta(x,t)$. The condition of stationarity of functional $F[\Phi,\eta]$ is equivalent to the non-homogeneous, nonlinear water-wave problem; see, e.g., Witham (1974). More precisely, the variational equation $\delta_\Phi F=0$ models the *water-wave kinematics*, while the variational equation $\delta_\eta F=0$ models the *water-wave dynamics* (Bernoulli's integral). One important property of the above variational formulation is that it can be used, in conjunction with various alternative representations of the wave potential $\Phi(x,z,t)$, to derive equivalent reformulations of the problem. One such possibility studied by the authors (Athanasoulis & Belibassakis 2002, Belibassakis & Athanasoulis 2006), is based on a *local-mode series expansion* of the wave potential in variable bathymetry regions with the following general form:

$$\Phi(x,z,t)=\sum_{n=-2}^{\infty}\varphi_n(x,t)\,Z_n\big(z,h(x),\eta(x,t)\big)\ , \tag{3}$$

where $Z_n(z,h,\eta)$ denotes the vertical structure of each mode (parametrically dependent on the ends of the vertical interval, see Fig.1), and $\varphi_n(x,t)$ its horizontal amplitude. All vertical functions are normalized, i.e. $Z_n(x,z=\eta)=1$, and thus, the sum of all modal amplitudes equals to the velocity potential on the free-surface,

$$\Phi(x,z=\eta;t)=\sum_{n=-2}\varphi_n(x;t)Z_n(x,z=\eta)=\sum_{n=-2}\varphi_n(x;t)=\varphi(x;t), \tag{4}$$

The vertical structure of the first two modes $Z_{-2}(z,h,\eta)$ and $Z_{-1}(z,h,\eta)$ is represented by low order z-polynomials. The term $\varphi_{-2}Z_{-2}$ is the *free-surface mode*, the term $\varphi_{-1}Z_{-1}$ is the *sloping-bottom mode* and the rest of the terms have the form

$$Z_n(z,h,\eta)=\frac{\cos\left[k_n(z+h)\right]}{\cos\left[k_n(\eta+h)\right]},\quad n=0,1,2,3,...\ , \tag{5}$$

where the z-independent quantities $k_n=k_n(h,\eta)$, $n=0,1,2...$, are the eigenvalues of regular Sturm-Liouville problems, formulated in the vertical interval $-h(x)<z<\eta(x,t)$, and obtained as the imaginary (n=0) and the real positive (n=1,2,3…) roots of the dispersion relation,

$$\mu_0+k_n\tan\left[k_n(h+\eta)\right]=0\,. \tag{6}$$

The term $\varphi_0 Z_0$ is the *propagating mode*, and the infinite terms $\varphi_n Z_n$, $n=1,2...$, are the *evanescent modes.* The (numerical) parameters μ_0, $h_0>0$ are positive constants, not subjected to any a-priori restrictions. A detailed proof concerning the above expansion can be found in Belibassakis & Athanassoulis (2006).

The main part of the above local-mode representation, consisted of the terms $n=0,1,2...$, is compatible with the standard eigensolutions of the linearised water-wave theory, at the local depth. The sloping-bottom mode (n=-1) has been first introduced by the authors (Athanasoulis & Belibassakis 1999) in order to consistently satisfy the bottom boundary condition on the non-horizontal parts of the seabed. Also, the free-surface term (n=-2) has been similarly introduced in order to enable the satisfaction of the non-linear boundary conditions on the free-surface. An important property of the above expansion is that it is rapidly converging. Thus, a small number of modes are enough for a precise numerical solution, provided that the first three terms (the free-surface, the sloping-bottom and the propagating ones) are included in the local-mode series, for various water depths, ranging from intermediate to shallow wave conditions.

The modal series (3) used in the variational equation (2) results to a *nonlinear Coupled-Mode System* (Athanassoulis & Belibassakis 2006, 2007) with respect to the wave potential on the free-surface $\varphi(x,t)$, the free-surface elevation $\eta(x,t)$ and the mode amplitudes $\varphi_n(x,t)$:

$$-\frac{\partial\eta}{\partial t}=\frac{\partial}{\partial x}\left((\eta+h)\frac{\partial\varphi}{\partial x}\right)+\sum_{n=-2}^{\infty}\left(\hat{A}_{-2n}(\eta)\frac{\partial^2\varphi_n}{\partial x^2}+\hat{B}_{-2n}(\eta)\frac{\partial\varphi_n}{\partial x}+\hat{C}_{-2n}(\eta)\varphi_n\right)\ , \tag{7a}$$

$$-\frac{\partial\varphi}{\partial t}-g\eta=N\ , \tag{7b}$$

where the nonlinear operator N is defined as follows:

$$N=\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2+\sum_{n=-2}^{\infty}\left(\left[W_n\right]_{z=\eta}\varphi_n\frac{\partial\eta}{\partial t}\right)-\sum_{\ell=-2}^{\infty}\sum_{n=-2}^{\infty}\left(a_{\ell n}^{(0,2)}(\eta)\varphi_\ell\frac{\partial^2\varphi_n}{\partial x^2}+b_{\ell n}(\eta)\varphi_\ell\frac{\partial\varphi_n}{\partial x}+c_{\ell n}(\eta)\varphi_\ell\varphi_n\right). \tag{7c}$$

In addition, the two-equation coupled system (7) is subjected to the constraints imposed by the following equations:

$$\sum_{n=-2}^{\infty}\left(A_{-2n}(\eta)-A_{mn}(\eta)\right)\frac{\partial^2\varphi_n}{\partial x^2}+\left(B_{-2n}(\eta)-B_{mn}(\eta)\right)\frac{\partial\varphi_n}{\partial x}+\left(C_{-2n}(\eta)-C_{mn}(\eta)\right)\varphi_n=0,\quad m=-1,0,1,2..,\qquad (8a)$$

$$\sum_{n=-2}\varphi_n(x;t)=\varphi(x;t),\qquad (8b)$$

which are shown to be equivalent to the kinematical subproblem materializing the DtN map, associated with the calculation of the wave potential in the whole domain *D*, given the instantaneous values of the free-surface potential (φ) and the free-surface elevation (η), and satisfying the bottom boundary condition. The coefficients $\hat{A}_{mn}(\eta)$, $\hat{B}_{mn}(\eta)$, $\hat{C}_{mn}(\eta)$ and $a_{mn}^{(0,2)}(\eta)$, $b_{mn}(\eta)$, $c_{mn}(\eta)$ are dependent on the free-surface elevation, and are defined in terms of the vertical modes; see Athanassoulis & Belibassakis (2006, 2007). An important property of the above nonlinear CMS is that various simplified models could be recovered as limiting forms. For example, keeping only the propagating mode in the vertical expansion and linearising the coupled-mode equations, the classical *mild-slope* model is obtained, see, e.g., Dingemans (1997). If the evanescent modes are also retained, an *extended mild-slope* model is obtained, see, e.g., Massel (1993). Detailed results concerning the dispersion characteristics of the linearised system are presented in Belibassakis & Athanassoulis (2006, Sec.7), where it is shown that retaining the first few terms in the series (up to 5 modes) is sufficient for numerical convergence to the exact result, for an extended range of wave frequencies, ranging from shallow to deep water-wave conditions. Finally, if we keep only the quadratic vertical mode and retain up to second-order terms in the present CMS, a *Boussinesq-type* model (see, e.g., Liu 1995) is obtained.

3 NUMERICAL SOLUTION OF THE NONLINEAR CMS

In the case of a general strip the numerical solution of the CMS is obtained by truncating the local-mode series (3) to a finite number of terms (modes), and using 2nd-order finite differences to approximate spatial derivatives. Time integration of the discrete system is obtained by an implicit low-order Runge-Kutta scheme. Discrete boundary conditions are obtained by using second-order forward and backward differences to approximate the horizontal derivatives at the ends of the horizontal interval. Numerical results of wave-motion starting from rest and excited by incident wave systems, propagating over flat bottom and shoals connecting two strips of constant (but possibly different) depths, can be found in Belibassakis & Athanassoulis (2006). In this case the derivation of steady traveling solutions of the above CMS is important not only for comparison and validation, but also for the consistent initialization of the coupled-mode system in the case of time evolution problems in non-homogeneous environments.

The problem of derivation of steady travelling wave solutions in constant but arbitrary depth h, characterised by the (unknown) wave celerity *c*, has been examined in Athanassoulis & Belibassakis (2006, 2007), using the transformation $\varphi(x;t)=\varphi(\xi)$, $\varphi_n(x;t)=\varphi_n(\xi)$, $\eta(x,t)=\eta(\xi)$, $\xi=x-ct$ (and thus, $\partial/\partial t=-c\partial/\partial x$), which enables us to put the CMS (7) to its time independent form:

$$c\,L_1(\mathbf{u})+L_0(\mathbf{u})=N(\mathbf{u}),\qquad (9)$$

where $\mathbf{u}=\begin{cases}\eta(x)\\ \varphi(x)\end{cases}$, $L_1(\mathbf{u})=\begin{cases}\partial\eta/\partial x\\ \partial\varphi/\partial x\end{cases}$, $L_0(\mathbf{u})=\begin{cases}0\\ -g\eta\end{cases}$, and the components of the nonlinear operator $N(\mathbf{u})$ are defined by the right-hand side of Eqs. (7a) and (7b). We note here that due to the fact that in the present case the bottom is horizontal ($h'=dh/dx=0$), the sloping-bottom mode identically vanishes ($\varphi_{-1}=0$), and Eq. (8a) for *m*=-1 is dropped. Given the length λ of the periodic cell (the wavelength) and the depth *h,* steady travelling wave solutions are numerically constructed by calculating the free-surface elevation $\eta(x)$, the modes $\varphi_n(x)$, $n=-2,0,1,2...$, the wave potential $\Phi(x,z)$, and the wave speed *c*. We remark here that a nontrivial solution should contain (at least) one crest. In order to obtain solution of mode type I (one crest and one trough per periodic cell) we need to specify the location of the crest. Without loss of generality, we suppose that $\eta'(x=x_*)=0$, at a given point x_* within the periodic cell. The latter condition introduces an additional algebraic constraint, that can be considered to be equivalent to a "nonlinear dispersion" relation. The system (10) is iteratively solved by: (i) guessing initial $c_0, \mathbf{u}_0$ (e.g., from a linear model), (ii) calculating $\mathbf{u}_{k+1} = \left(c_k L_1 + L_0\right)^{-1} N(\mathbf{u}_k)$, and c_{k+1} from the above constraint (concerning the position of the crest), and (iii) iterating until convergence: $\left\|\mathbf{u}_{k+1}-\mathbf{u}_k\right\| + \left|c_{k+1}-c_k\right| < \text{tolerance}$.

Detailed numerical results are presented in Athanassoulis & Belibassakis (2006), where it is shown that the present CMS provides solutions fully compatible with high-order Stokes theory, in intermediate water depth, and with non-linear cnoidal theory in the long-wave regime. It is also shown that the present model provides reasonable and useful results at all points of the parameter domain (see Fig.2, left), from low non-linearity up to the breaking limit $H/h = 0.142\tanh(2\pi h/\lambda)$, where *H* is the waveheight, as well as in the regime of Ursell number $U=(H/h)(\lambda/h)^2 \approx 8\pi^2$, where the usual Stokes expansion fails and Boussinesq models are applicable. One example, characterised by shallow water depth (*λ/h*=20) and moderate nonlinearity (*H/h*=0.3) is presented in Fig. 2. The initial guess, based on the linearised (time-harmonic) solution, is plotted in the right part of the figure, as well as the consecutive iterations. In this case, the final convergent solution of the present CMS is fully compatible with the prediction by the non-linear cnoidal theory, Fenton (1990), and the phase speed (*c*) of the wave is found to be 6% higher than one predicted by linear theory.

4. THE WEAKLY DISSIPATIVE BOUNDARY LAYER MODEL

The effects of viscosity in the bottom boundary layer are of the order $O\left(\mathrm{Re}^{-1/2}\right)$ (see, e.g., Mei, 1983, Debnath 1994), where a possible definition of the Reynolds number in the case of (nearly harmonic) wave flows is $\mathrm{Re} = c\lambda/\nu$ with *c* and *λ* the characteristic values of phase speed and wavelength, respectively, and *ν* is the liquid viscosity. On the basis of linearised wave theory an alternative definition of the Reynolds number in terms of the wave frequency is $\mathrm{Re} = 2\pi g^2/\nu\omega^3$. As suggested by various authors (e.g. Liu 2004, Dutykh & Dias 2007a,b), in the case of laminar oscillating boundary layer flow, the eddy viscosity should be used instead of the kinematic viscosity. (We note here that, for water, the eddy viscosity is of the order $10^{-3} m^2/s$, i.e. 10^3 times greater than the kinematic viscosity). Moreover, the effects of viscosity associated with the free-surface boundary layer are $O\left(\mathrm{Re}^{-1}\right)$ and with the bulk of the fluid $O\left(\mathrm{Re}^{-3/2}\right)$, respectively.

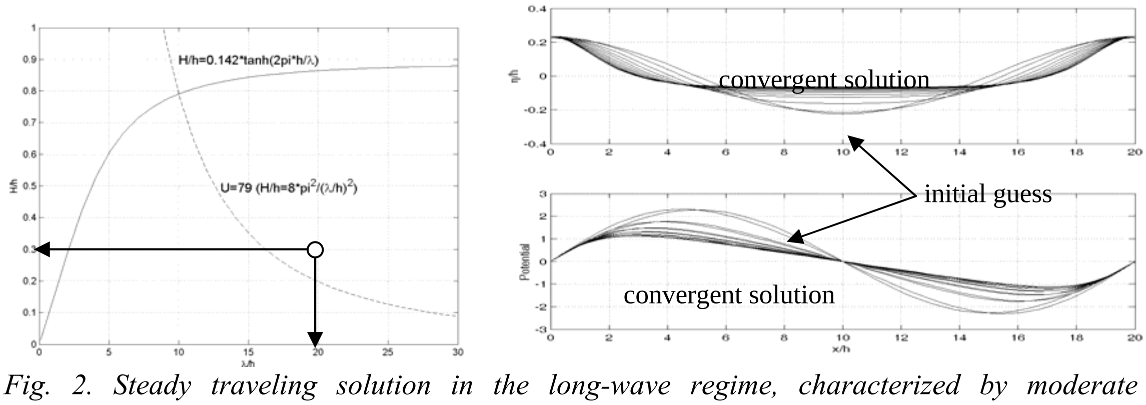

Fig. 2. Steady traveling solution in the long-wave regime, characterized by moderate nonlinearity (λ/h=20, H/h=0.3, c/c₀=1.06). Left: position on the nonlinearity-shallowness parameter space. Right: Initial guess and final converged solution.

Aiming to derive an enhanced version of the present CMS in constant depth, taking into account the effects of viscosity up to the leading-order, we focus herein on the laminar bottom boundary layer of waves propagating over a flat (or very mildly sloped) bottom. Using the standard scaling from physical $(x,\ z,\ t,...)$ to non-dimensional variables $(\tilde{x},\ \tilde{z},\ \tilde{t},...)$

$$\tilde{t} = c\,t/\lambda, \quad \tilde{x} = x/\lambda\,, \quad \tilde{z} = z/h\,, \tag{10}$$

and the following one concerning wave flow velocity (u,w), pressure (p) and free-surface elevation $\eta(x;t)$:

$$\tilde{u} = (1/\varepsilon)\ u/c, \quad \tilde{w} = (\kappa/\varepsilon)\ w/c, \quad \tilde{p} = p/(\rho g H/2), \quad \tilde{\eta} = \eta/(H/2)\,, \tag{11}$$

where H denotes the (local) the waveheight, in the simple case of 2D vertical wave motion, the equations governing the flow (continuity and x- and z- momentum equations) become,

$$\kappa^2 \tilde{u}_x + \tilde{w}_z = 0\,, \tag{12}$$

$$\tilde{u}_t + \varepsilon \tilde{u}\tilde{u}_x + \left(\varepsilon/\kappa^2\right)\tilde{w}\tilde{u}_z = -\left(\mathrm{Fr}^{-2}\right)\tilde{p}_x + \left(\mathrm{Re}^{-1}\right)\left(\tilde{u}_{xx} + \kappa^{-2}\tilde{u}_{zz}\right), \tag{13}$$

$$\varepsilon \tilde{w}_t + \varepsilon \tilde{u}\tilde{w}_x + \left(\varepsilon^2/\kappa^2\right)\tilde{w}\tilde{w}_z = -\left(\mathrm{Fr}^{-2}\right)\varepsilon\,\tilde{p}_z - \left(\mathrm{Fr}^{-2}\right) + \left(\varepsilon\,\mathrm{Re}^{-1}\right)\left(\tilde{w}_{xx} + \kappa^{-2}\tilde{w}_{zz}\right)\ . \tag{14}$$

In the above equations, the lower indices denote differentiation with respect to the scaled vatiables $(\tilde{x},\ \tilde{z},\ \tilde{t})$, $\mathrm{Fr} = c/\sqrt{gh}$ is the Froude number based on the local wave celerity, and $\varepsilon=(H/2)/h$ and $\kappa=h/\lambda$ denote the wave nonlinearity parameter and the shoaling parameter, respectively. (The present nonlinearity parameter ε is related with the standard one $\in$ used in Stokes theory by $\in = 2\pi\left(\mathrm{H}/2\right)/\lambda = 2\pi\,\varepsilon\,\kappa$). A similar system has been derived and studied by Liu & Orfila (2004), in the long-wave regime. In that case the coefficient Fr^{-2} in front of the first terms in the rhs of the above momentum equations is missing, since the linearised shallow water approximation of the phase speed has been used for scaling ($c = \sqrt{gh}$). Thus, the present model is fully compatible with the one derived by Liu & Orfila (2004, Eqs. 2.3, 2.4, 2.5), and reduces asymptotically to the latter in the shallow water approximation.

The scaled z-variable ranges in $-1 \le \tilde{z} \le \left(H/2h\right)\tilde{\eta}\left(x;t\right)$. Following the standard approach, we introduce the stretched coordinate $S = \kappa\sqrt{\mathrm{Re}}\ \left(\tilde{z}+1\right)$, ranging from zero to very high value (and asymptotically $0 < S < \infty$), and the usual boundary layer expansions

$$\tilde{u} = \tilde{\Phi}_{,x} \ + \ \tilde{u}_0 \ + \ \left(\mathrm{Re}^{-1/2}\right)\tilde{u}_1 \ + ...\ , \tag{15a}$$

$$\tilde{w} = \tilde{\Phi}_{,z} \ + \ \ + \ \left(\kappa\ \mathrm{Re}^{-1/2}\right)\tilde{w}_1 \ + ...\ , \tag{15b}$$

where $\left(\tilde{\Phi}_x, \tilde{\Phi}_z\right)$ denote the exterior potential flow velocity components near the bottom (in scaled variables), and $\left(\tilde{u}_k, \tilde{w}_k\right)$ are the velocity boundary-layer correction terms at various orders ($k = 0,1,2...$). In the case of waves propagating in a horizontal strip ($h' = 0$), the analysis has shown that the zero order vertical velocity term vanishes ($\tilde{w}_0 = 0$), and thus the latter term is missing from the expansion (15b). However, this particular term will come into play again, when considering bottom-boundary layer effects of waves propagating in variable bathymetry regions. The study of the general, sloping-bottom problem is more complicated by the fact that the stretched variable needs to be defined with direction normal to the bottom contour and is left to be considered in future work.

Using the expansions (15) in Eqs. (12), (13), (14), we obtain at the leading order from the continuity equation (12) the Laplace equation for the wave potential ($\Delta\Phi = 0$), that models the exterior irrotational flow (where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$ denotes the 2D Laplacian on the vertical strip). In the case of a flat bottom **(** $h' = 0$), we obtain at the next order $O\left(\kappa^2\right)$ the boundary layer continuity and momentum equations, respectively:

$$\tilde{u}_{0x} + \tilde{w}_{1,S} = 0 \ , \qquad \tilde{u}_{0t} + \varepsilon\left(\tilde{u}_0 \, \tilde{u}_{0x} + \tilde{w}_1 \tilde{u}_{0S}\right) = \tilde{u}_{0,SS} \ , \qquad \tilde{p}_{,S} = 0 \ . \tag{16}$$

Keeping terms up to zero-order in wave nonlinearity (ε) in the x-momentum equation, we obtain the standard parabolic equation for the horizontal velocity correction $(\tilde{u}_0)$ in the boundary layer:

$$\tilde{u}_{0,t} + \tilde{u}_{0,SS} = 0 \ , \tag{17}$$

with boundary conditions

$$\tilde{u}_0 = -\frac{1}{\varepsilon c}\frac{\partial \Phi}{\partial x}\bigg]_{z=-h} , \qquad \tilde{w}_1 = -\frac{\kappa}{\varepsilon c}\frac{\partial \Phi}{\partial z}\bigg]_{z=-h} , \qquad \text{at} \quad S = 0 \ , \tag{18a}$$

$$\tilde{u}_0 = 0, \qquad \tilde{w}_1 = 0, \qquad \text{at} \quad S = \infty \ . \tag{18b}$$

The boundary condition (18b) can be practically applied to a height above the solid bottom of the order of the thickness of the boundary layer. In the case of laminar flow the latter is estimated to be: $\delta = 4 \div 5 \sqrt{\nu\lambda / c}$, and thus, the boundary conditions (18b) are approximately enforced at $S = \kappa \, \delta \, \sqrt{\mathrm{Re}} = O(\kappa)$ (instead of $S = \infty$). Using the Green's function of the parabolic equation, we finally obtain from the solution ($\tilde{u}_0$) of the above problem, in conjunction with the continuity equation (16a), the following expression concerning the distribution of the vertical velocity at the bottom (expressed in physical variables):

$$w_1 = \frac{\partial \Phi}{\partial z}\bigg]_{z=-h} = -\sqrt{\frac{\nu}{\pi}} \int_{\tau=0}^{\tau=t} \Phi_{,xx}\left(x, z=-h; t\right) \left[t-\tau\right]^{-1/2} d\tau \quad , \quad \text{at} \qquad z = -h \qquad . \tag{19}$$

The above relation provides the mass imbalance generated by the boundary layer flow near the solid bottom surface, which can be compensated only by the sloping-bottom mode (n=-1 term in our formulation). We recall here that the latter term has been set equal to zero ($\varphi_{-1} = 0$) when treating the exterior irrotational wave flow over a flat bottom (cf. Sec.3). Combining Eq. (19) and the definition of the sloping-bottom mode (see Athanassoulis & Belibassakis 2002, Belibassakis & Athanassoulis 2006), we obtain the following additional equation

$$-h_0 \sqrt{\frac{\nu}{\pi}} \int_{\tau=0}^{\tau=t} \sum_{n=-2} \frac{\partial^2 \left(\varphi_n\left(x;t\right) Z_n\left(z=-h;z\right)\right)}{\partial x^2} \left[t-\tau\right]^{-1/2} d\tau = \varphi_{-1}\left(x;t\right) \ . \tag{20}$$

Eq. (20) models the effects associated with the mass flux generated at the bottom by the laminar boundary layer and enhances the present CMS by completing the set of the kinematical constraints defined by Eqs. (8) in the examined ($h' = 0$) case.

As a numerical example, we present in Fig.3 the structure of the calculated wave field for the same case as previously considered in Fig. 2, by using equipotential lines. In Fig. 4 (left column of subplots) we present the vertical profiles of the calculated boundary-layer velocities $\left(u_0(z), w_1(z)\right)$ within the bottom boundary layer, corresponding to the various horizontal positions (or phases of the wave) indicated in Fig. 3 by using vertical dashed lines. Similarly, in right column of subplots of Fig. 4 we present the vertical profiles of the total wave velocities $\left(u(z), w(z)\right)$ all over the water column. Numerical evidence and first comparisons with other methods suggest that the present model provides reasonable solutions in constant, but arbitrary depth, not only in the shallow water regime but also in intermediate water depth, taking properly into account the effects of laminar boundary layer. Extensive comparisons with other theories and experimental data aiming to validate the enhanced CMS will be presented elsewhere.

5 CONCLUSIONS

The non-linear CMS has been obtained without any assumptions concerning the nonleariry and the vertical structure of the wave potential, being thus, equivalent with the complete water-wave formulation. For water waves propagating over a flat bottom, the above formulation is linked with laminar bottom boundary layer equations, permitting the investigation of viscous effects on wave propagation up to leading-order. In this case, the present method is shown to represent

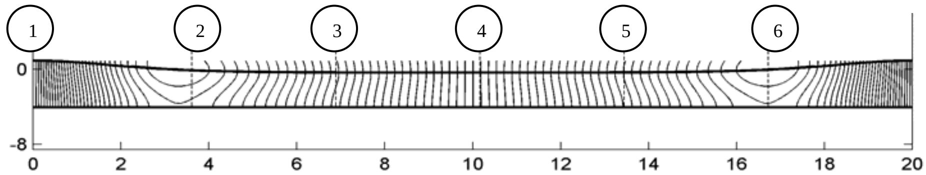

Fig 3. Contour plot of the steady traveling wave as obtained by the solution of the present CMS in the long-wave regime, characterized by moderate nonlinearity (λ/h=20, H/h=0.3, c/c_0=1.06).

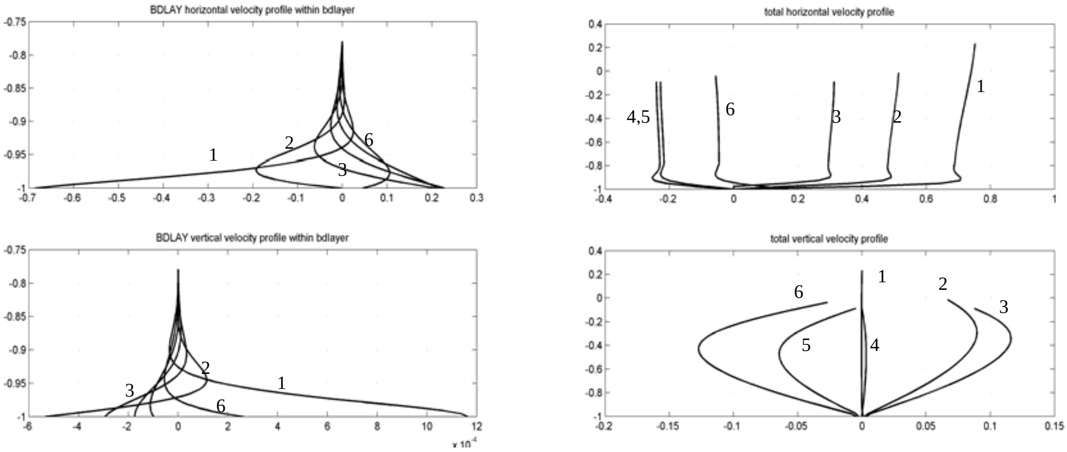

Fig. 4. (Left) Profiles of horizontal (top) and vertical (bottom) boundary layer velocity within the boundary layer (near the bottom) at the six horizontal positions shown in Fig. 3. (Right) Corresponding profiles of horizontal (top) and vertical total flow velocities under the wave.

well the structure of laminar bottom boundary layer, permitting the accurate estimation of viscous damping of progressive waves in intermediate and shallow water depth conditions. Future work is directed to the extension of the present weakly dissipative CMS to waves propagating in general bathymetry regions.

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