Homotopy analysis method for some nonlinear water wave problems*

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ABSTRACT: In this paper, an analytic technique, namely the homotopy analysis method (HAM), is applied to the nonlinear water wave problems.In order to verify the validity of the HAM for the nonlinear water waves problems,two numerical tests have been carried out.One case is one dimensional nonlinear shallow water wave (KdV equation),the other case is nonlinear deep water waves (Schrödinger equation).The approximate solution of KdV equation is obtained using homotopy analysis method,and agrees well with the exact solution. Then,we apply it to solve nonlinear Schrödinger equation. Such a kind of explicit, analytic solutions are useful for analyzing periodic wave groups and envelope solitary gravity waves.

KEY WORDS: nonlinear Schrödinger equation;periodic wave groups, explicit analytic solution;envelope solitary gravity waves, Homotopy analysis method.

1 INTRODUCTION

Recent years saw rapidly increasing number of papers on nonlinear water waves due to their potentially devastating effects on ships and offshore structures.Direct analytical solution of the N-S equations or Euler equations is paramountly difficult, various simplified nonlinear theories for water waves have been proposed, among which the well known ones are KdV equations for shallow water (Korteweg;1895) and the nonlinear Schrödinger (NLS) equations for deep water waves (Zabusky; 1965).

Most of previous theoretical studies of nonlinear water waves are based on perturbation methods applicable to cases where a small parameter is present. For strngly nonlinear problems, howevere, pertubation methods fail to give accurate prediction. Efforts to seek more powerful analytical tool for solving strongly nonlinear poorblmes gave rise to some effective methods. Examplesa are Adomian decomposition method, Lyapunov artificial-small-parameter method, Delta expansion method and homotopy Analysis method [HAM](Liao;2003). HAM is attractive in the sense that it is independent of whether a small parameter is present or not. Thus, the HAM is valid for nonlinear problems, either weak or strong.

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2 NONLINEAR SHALLOW WATER WAVES GOVERNED BY KDV EQUATION

We discuss solitary wave solution of the KdV equation.The KdV equation describes motions of long waves in shallow water under gravity in one dimensional flow.

2.1 *Mathematical formulation*

Consider the initial value problem associated with the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, x \in \mathbb{R}, \tag{1}$$

$$u(x,0) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} \tag{2}$$

where $u = u(x,t)$ is a differentiable function.In addition,we assume that the solution $u(x,t)$,along with its derivatives ,tends to zero as $|x| \longrightarrow \infty$. The exact solution is found to be(Wazwaz;2001)

$$u(x,t) = -2\frac{k^2 e^{k(x-k^2t)}}{(1+e^{k(x-k^2t)})^2}. \tag{3}$$

From Eq. (1), we define the nonlinear operator

$$\mathcal{N}[U(x,t;q)] = \frac{\partial U(x,t;q)}{\partial t} - 6U(x,t;q)\frac{\partial U(x,t;q)}{\partial x} + \frac{\partial^3 U(x,t;q)}{\partial x^3}. \tag{4}$$

According to the initial condition denoted by (2), it is natural to choose

$$u_0(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}. \tag{5}$$

We choose the auxiliary linear operator

$$\mathcal{L}[U(x,t;q)] = \frac{\partial U(x,t;q)}{\partial t}. \tag{6}$$

with the property $\mathcal{L}[C_1] = 0$, where C_1 is coefficient.

The homotopy analysis method is based on a continuous transform $U(x,t;q)$, as the embedding parameter q increases from 0 to 1, $U(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$.

To ensure this, let $h \neq 0$ denote an auxiliary parameter, $q \in [0,1]$ an embedding parameter. We have the zeroth-order deformation equation

$$(1-q)\mathcal{L}[U(x,t;q) - U_0(x,t)] = qh\mathcal{N}[U(x,t;q)], \tag{7}$$

subject to the initial condition

$$U(x,0;q) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}. \tag{8}$$

Thus, $U(x,t;q)$ can be expanded in the Maclaurin series with respect to q in the form

$$U(x,t;q) = U(x,t;0) + \sum_{m=1}^{+\infty} u_m(x,t)q^m. \tag{9}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m U(x,t,;q)}{\partial q^m}|_{q=0}. \tag{10}$$

Note that the zeroth-order deformation Eq.(7) contains the auxiliary parameter h, so that $U(x,t;q)$ is dependent on h. Assuming that h is so properly chosen that the series Eq.(9) is convergent at $q=1$, we obtain from Eq.(9) that

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t). \tag{11}$$

Fore the sake of simplicity, introduce

$$\vec{u}_m = \{u_1, u_2, \ldots, u_m\}. \tag{12}$$

We differentiate the zeroth-order deformation Eq.(7) m times with respect to q, then set $q=0$. Dividing the obtained equation by factorial $m!$, we get the so-called mth-order deformation equation:

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hR_m(\vec{u}_{m-1}) \tag{13}$$

subject to the initial condition

$$u_m(x,0) = 0, (m \geq 1) \tag{14}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - 6\sum_{j=0}^{m-1} u_j u_{(m-1-j)x} + \frac{\partial^3 u_{m-1}}{\partial x^3}, \tag{15}$$

and

$$\chi_m = \begin{cases} 0 & \text{when } m \leq 1, \\ 1 & \text{when } m > 1. \end{cases} \tag{16}$$

2.2 *Results and analysis*

We now successively obtain the solution to each high order deformation equation:

$$u_m(x,t) = \chi_m u_{m-1} + h\int_0^t R_m(\vec{u}_{m-1})dt + C_1 \tag{17}$$

where C_1 is an integral constant to be determined from the initial condition (8). The solution in a series form is given by

$$u(x,t) = -8\,\frac{e^{2\,x}}{\left(1+e^{2\,x}\right)^2} + 64\,\frac{he^{2\,x}\left(-1+e^{2\,x}\right)t}{\left(1+e^{2\,x}\right)^3} + \ldots \tag{18}$$

The above series (18) contains the auxiliary parameter h , which influences the convergent rate and convergence region . To ensure that the series converge, we first focus on how to choose a proper value of h . We can first investigate the influence of h on the series of $u(0.1,1)$ by means of the so called h-curve, that is, a curve of $u(0.1,1)$ versus h. As pointed out by Liao , the valid region of h is a horizontal line segment. Thus, the valid region of h in this case is $-1.8 < h < -0.5$, as shown in Fig.1. As proved by Liao , the solution series (18) must be exact solution, as long as it is convergent. For example, our analytic solution converges when $-0.2 < t < 0.2$ and $h=-1$, as shown in Fig.2,compared with the exact solution.Besides, much more accurate results can be obtained by means of

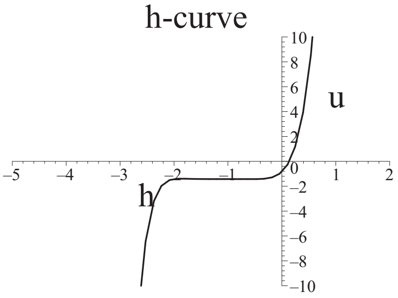

Figure 1: The h-curve of $u(0.1,1)$ at 8th-order approximation.

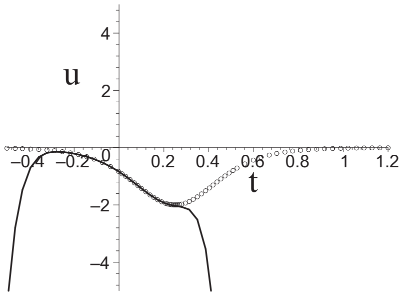

Figure 2: Comparison of the exact solution with the 8th-order HAM solution of $u(1,t)$ Open circles: exact solution; solid line: HAM solution.

the so-called homotopy-Padé technique .The corresponding homotopy-Padé approximation of $u(1,t)$ converge to the exact result in the whole field of time, as shown in Fig.3. The approximation by using homotopy analysis method and the exact solution are graphically the same ,as shown in Fig.4. Fig.5 presents u profile along the spatial variable x ,respectively when $t=0$,$t=0.1$,$t=0.15$ and $t=0.2$,which means the wave is a solitary wave.

3 DEEP-WATER WAVE TRAINS

3.1 *The nonlinear Schrödinger equation*

The nonlinear Schrödinger equation is

$$i(\psi_t + C_g\psi_x) + \mu\psi_{xx} + \nu|\psi|^2\psi = 0. \tag{19}$$

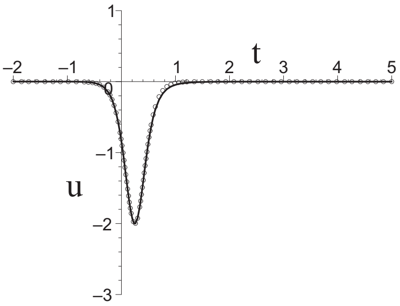

Figure 3: Comparison of the exact solution with the [1,6] Homotopy -Padé solution of $u(1,t)$ Open circles: exact solution; solid line: H-P solution.

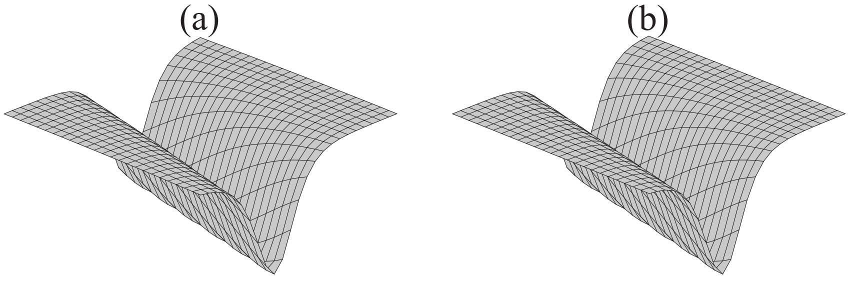

Figure 4: Comparison of the exact solution with the HAM solution of $u(x,t)$, when $h=-1$,(a)HAM solution, (b)exact solution .

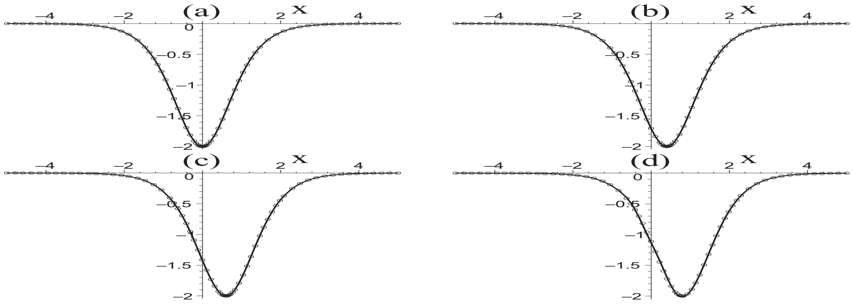

Figure 5: Comparison of the exact solution with the HAM solution at 8th-order when $h=-1$,$t=0, t=0.1, t=0.15$ and $t=0.2$,respectively.Open circles: exact solution ; solid line:HAM approximation.

which appears in optics and the theory of water waves, and also be considered as a second quantized Schrödinger field. In optics, the nonlinear Schrödinger equation occurs in the Manakov system, a model of wave propagation in fiber optics. The function $\psi(x,t)$ represents a wave and the nonlinear Schrödinger equation describes the propagation of the wave through a nonlinear medium.In the theory of water waves, the nonlinear Schrödinger equation describes the evolution of the envelope of modulated wave groups.In 1968, Zakharov described the Hamiltonian structure of water waves and showed that for slowly-modulated wave groups, the wave amplitude satisfies the nonlinear Schrödinger equation, approximately. In this paper,we take the nonlinear Schrödinger equation as a deep water wave model.Namely,the NLS equation describes the dynamics of a deep-water wave train which propagates in the x direction. This equation solves the Cauchy problem, namely, given the complex envelope at some initial time $t=0$, $\psi(x,0)$, Eq.(19) evolves the dynamics for all space and time, $\psi(x,t)$.

For waves on the water surface of deep water, the important coefficients for the nonlinear Schrödinger equation are: $C_g=\frac{1}{2}\frac{\omega_0}{k_0}=\frac{1}{2}\frac{L_0}{T_0}, \mu=-\frac{\omega_0}{8k_0^2}, \nu=-\frac{1}{2}\omega_0 k_0^2$ and $\lambda=\sqrt{\frac{\nu}{2\mu}}=\sqrt{2}k_0^2$. Here the subscript "0" denotes the carrier wave which is modulated by the function $\psi(x,t)$: k_0 is the wave number, ω_0 is the carrier wave frequency, L_0 is the carrier wave length and T_0 denotes the carrier wave period. The complex field $\psi(x,t)$, as appearing in the nonlinear Schrödinger equation, is related to the amplitude and phase of the water waves. Consider a slowly-modulated carrier wave with water surface elevation $\eta(x,t)$ of the form

$$\eta(x,t)=Re[\psi(x,t)e^{i(k_0x-\omega_0t)}]/2. \tag{20}$$

Thus, the carrier $e^{i(k_0x-\omega_0t)}$ is modulated by the complex envelope, $\psi(x,t)$, as determined by Eq.(19) for some chosen initial condition, $\psi(x,0)$.

3.2 *The non-dimensional form of the NLS equation*

The simple transformation

$$u=\lambda\psi, T=\mu t, X=x-C_gt \tag{21}$$

allows Eq.(19) to be put into non-dimensional form. The non-dimensional NLS equation arises

$$iu_T+u_{XX}+2|u|^2u=0. \tag{22}$$

where λ is a scale factor.This simple form of NLS is often used for mathematical convenience.

3.3 *Mathematical formulation*

The initial condition

$$u(X,0)=g(X) \tag{23}$$

In the complex plane ,we obtain the following equation:

$$|u|^2=u\bar{u}. \tag{24}$$

Multiply both sides of Eq.(22),we gain the following equivalent equation:

$$u_T-iu_{XX}-2i|u|^2u=0. \tag{25}$$

4 RESULTS AND ANALYSIS

4.1 *Periodic groups*

For one case,the initial condition is

$$u(X,0)=e^{iX}. \tag{26}$$

According to the initial condition(26), it is straightforward that the initial approximation should be in the form

$$u_0(X,T)=e^{iX} \tag{27}$$

The Mth-order approximation is given by

$$u(X,T)\approx\sum_{m=0}^{M}u_m(X,T) \tag{28}$$

When $h=-1$, We gain the exact explicit solution

$$u(X,T)=\sum_{k=0}^{+\infty}u_k(X,T)=e^{iX}\sum_{n=0}^{+\infty}\frac{(iT)^n}{n!}=e^{i(X+T)} \tag{29}$$

To study the periodic solutions, we obtain a typical set of profiles displayed in Fig.6 for different values of wave number k_0.

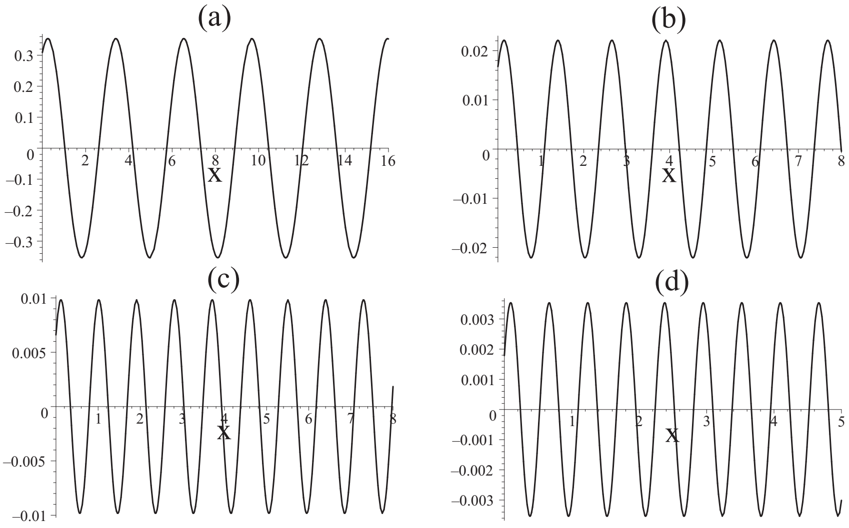

Figure 6: Profiles of the surface deformation η solution of (19) with $t = 0.1$ and k_0 taking the values 1 ,4 ,6 and 10,from a-d.

4.2 *Envelope solitary gravity waves*

For the other case,the initial condition is

$$u(X,0) = \mathrm{sechX}. \tag{30}$$

the initial approximation is chosen as

$$u_0(X,T) = \mathrm{sechX} \tag{31}$$

Substituting it into the high order deformation equation,it is easy to get the solution of the linear differential equation,especially by means of symbolic software such as Mathematica, Maple, MathLab, and so on. Its solution can be expressed in the form,when $h = -1$, We also can obtain the exact explicit solution

$$u(X,T) = \sum_{k=0}^{+\infty} u_k(X,T) = \sum_{n=0}^{+\infty} \frac{(iT)^n}{n!} \mathrm{sechX} = \mathrm{e}^{\mathrm{iT}} \mathrm{sechX}. \tag{32}$$

For example , the wave number of the carrier wave $k_0 = 4$,a hyperbolic secant (sech) envelope soliton for surface waves on deep water is shown in Fig.7.The envelope solitary wave retains its identities with the same shape as time marches. It should have remarkable stability properties.

5 CONCLUSION

An analytical study of periodic wave groups and envelope solitary gravity waves properties have been performed based on explicit analytic solutions of the NLS equation that describes the nonlinear evolution of deep-water wave trains.This indicates that the Homotopy Analysis Method is valid for periodic wave groups and envelope solitary gravity wave problems in nonlinear deep water.

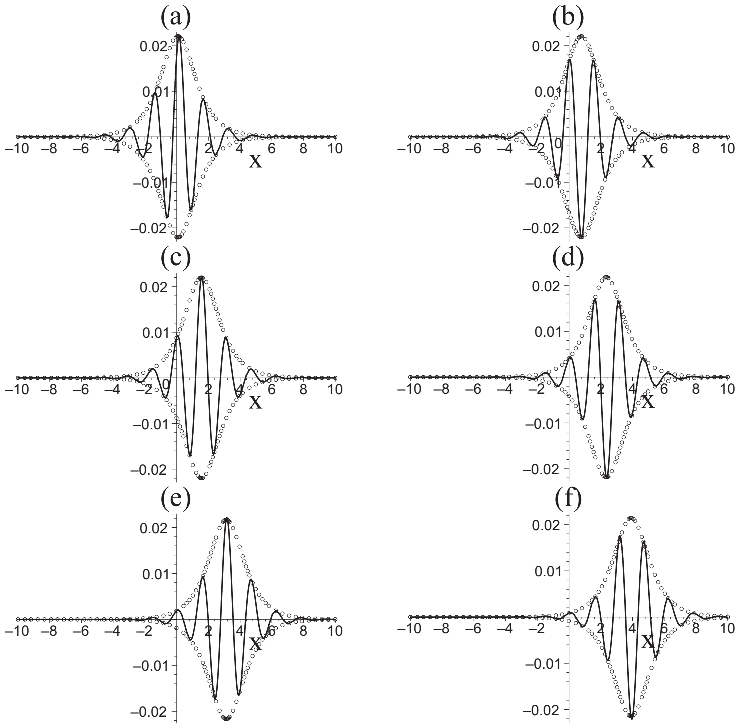

Figure 7: Evolution of envelop soliton and surface elevation when $t = 0.1, t = 1$, $t = 2, t = 3$,$t = 4$ and $t = 5$,respectively,from a-f .
Solid line: water waves; open circles: envelope soliton.

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